Homework 11

Sunday, October 8, 2023 2:13 PM

Math 381 - Discrete Mathematics
Maddie Brown
Due: October 10 2023

Assignment 11

Chapter 2.1

Required: 24, 32, 44, and Problems A and B written below.
Suggested: 33

Chapter 2.2

Required: 2, 4, 14
Suggested: 1, 3

Problem A Let $A = \{x \in \mathbb{R} \mid ax^2 + bx + c = 0$ for some integers $a, b,$ and c, with at least one of a, b, c nonzero$\}$. Let $B = \{x \in \mathbb{R} \mid px^2 + qx + r = 0$ for some rationals p, q, r with at least one of p, q, r nonzero$\}$.

1. Prove $2 \in A$ and $\sqrt{2} \in A$.
2. Give and example of a real number y that is not in A (you don't need to prove it).
3. Prove $A = B$.

Problem B Define set A by $A = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}$.

1. Give a geometric description of A.
2. Define a new "addition" $\diamond$ on this set according to the following rule:

$$(x, y) \diamond (z, w) = (xw + zy, wy) \quad \text{for} \quad (x, y) \in A \text{ and } (z, w) \in A$$

where the symbol + denotes regular addition. Show that the $\diamond$-addition of two elements of A is still in set A.

3. Find an element $(a, b) \in A$ such that $(a, b) \diamond (x, y) = (x, y)$ for every $(x, y) \in A$.
4. This "new addition" probably looks odd, but you have seen it before. What is it?

1

§2.1

24. Can you conclude that $A = B$ if A and B are two sets w/ the same power set?

We assume this to be true.

$$P(A) = P(B)$$

$A \in P(A)$
$A \in P(B)$
$A \subseteq B$
$B \in P(B)$
$B \in P(A)$
$B \subseteq A$
Since A is a subset of B and B is a subset of A, we know $A = B$ ☒

32. Suppose $A \times B = \emptyset$, where A and B are sets. What can you conclude?

The cartesian product is defined as the collection of ordered pairs in sets A and B. If $A \times B = \emptyset$ then A or B must be empty sets or they are both empty, since there is no pair to be matched w/ in the null set. ☒

44. Prove or Disprove that if A, B, and C are nonempty sets and $A \times B = A \times C$, then $B = C$.

$P \wedge Q \rightarrow R \equiv \neg R \rightarrow \neg(P \wedge Q)$ by Contrapositive
$\equiv \neg R \rightarrow \neg P \vee \neg Q$ by De Morgan's Law

Proof by Contrapositive:
Let $B \neq C$ s.t. $\{x \mid x \in B \wedge x \notin C\}$

If we let the element y be an element in the set A, s.t. $\{(y, x) \mid [(y, x) \in A \times B] \wedge [(y, x) \notin A \times C]\}$. So, $(A \times B \neq A \times C)$ or (A, B, or C is an empty set).
Therefore, our original statement, "if A, B, and C are nonempty sets and $A \times B = A \times C$, then $B = C$", is also true. ☒

§2.2

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school.

$A \cap B$

b) the set of sophomores at your school who are not taking discrete mathematics.

$A \cap \overline{B}$

c) the set of sophomores at your school who either are sophomores or are taking discrete mathematics.

$A \cup B$

d) the set of sophomores at your school who are either not sophomores or are not taking discrete mathematics.

$\overline{A} \cup \overline{B}$

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$

$A \cup B = \{a, b, c, d, e, f, g, h\}$

b) $A \cap B$

$A \cap B = \{a, b, c, d, e\}$

c) $A - B$

$A - B = \emptyset$

d) $B - A$

$B - A = \{f, g, h\}$

14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

$A \cap \overline{B} = \{1, 5, 7, 8\}$
$\overline{A} \cap B = \{2, 10\}$
$A \cap B = \{3, 6, 9\}$
So, $A = \{1, 3, 5, 6, 7, 8, 9\}$ and $B = \{2, 3, 6, 9, 10\}$
because $A = (A \cap \overline{B}) \cup (A \cap B)$ and $B = (\overline{A} \cap B) \cup (A \cap B)$. ☒

Problem A Let $A = \{x \in \mathbb{R} \mid ax^2 + bx + c = 0$ for some integers $a, b,$ and c w/ at least one of a, b, c nonzero$\}$. Let $B = \{x \in \mathbb{R} \mid px^2 + qx + r = 0$ for some rationals p, q, r w/ at least one of p, q, r nonzero$\}$.

1. Prove $2 \in A$ and $\sqrt{2} \in A$

$A = \{x \in \mathbb{R} \mid ax^2 + bx + c = 0$ for some integers $a, b,$ and c w/ at least one of a, b, c nonzero$\}$
Let $2 \in A$

$a(2)^2 + b(2) + c = 0$
$\Rightarrow 4a + 2b + c = 0$
Let $c = 0$ s.t.
$\Rightarrow 4a = -2b$
Let $a = 1$, $b = -2$
$\Rightarrow 4 = 4$ ✓
t.f. $2 \in A$ ☒

Let $\sqrt{2} \in A$

$a(\sqrt{2})^2 + b(\sqrt{2}) + c = 0$
$\Rightarrow 2a + \sqrt{2}b + c = 0$
Let $c = 0$
$\Rightarrow 2a = -\sqrt{2}b$
Let $a = 1$, $b = -\sqrt{2}$
$\Rightarrow 2 = 2$ ✓
t.f. $\sqrt{2} \in A$ ☒

2. Give an example of a real number y that is not in A (you don't need to prove it).
If we let y be equal to the real number $\emptyset$ this will not be in A since the condition at least one of a, b, c are nonzero will not be met. ☒

3. Prove $A = B$

$B = \{x \in \mathbb{R} \mid px^2 + qx + r = 0$ for some rationals p, q, r w/ at least one of p, q, r nonzero$\}$

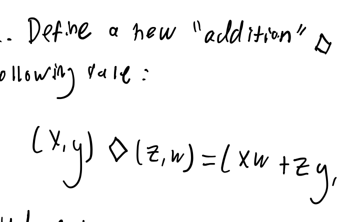
$A = B \equiv (A \subseteq B) \wedge (B \subseteq A)$
or $\forall x((x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A))$
$A \subseteq B$
Let $x \in A$ s.t. $ax^2 + bx + c$ for some integers $a, b,$ and c w/ at least one of a, b, c nonzero.

$ax^2 + bx + c = 0 \Rightarrow \frac{a}{d}x^2 + \frac{b}{d}x + \frac{c}{d} = 0$, for some $d \in \mathbb{Z}$, $d \neq 0$
this can be written as
$px^2 + qx + r = 0$ since p, q, and r are all rational numbers defined as $\frac{p}{q} = x$ where $p, q \in \mathbb{Z}$ and $q \neq 0$
So, B is true
$B \subseteq A$
Let $x \in B$ s.t. $px^2 + qx + r = 0$ for some rationals p, q, r w/ at least one of p, q, r nonzero.

$px^2 + qx + r = 0 \Rightarrow p(x^2 + \frac{q}{p}x + \frac{r}{p}) = 0$
It is not true since $\frac{q}{p}$ doesn't need to equal an integer i.e. $\frac{3}{2} = 1.5$, whereas this is required for set A.
Therefore $A \subseteq B$ but $B \not\subseteq A$. ☒

Problem B Define set A by $A = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}$

1. Give a geometric description of A.



The set A represents all points on an xy-plane other than those where $y = 0$. ☒

2. Define a new "addition" $\diamond$ on this set according to the following rule:

$$(x, y) \diamond (z, w) = (xw + zy, wy) \text{ for } (x, y) \in A \text{ and } (z, w) \in A$$

where the symbol + denotes regular addition. Show that the $\diamond$-addition of two elements of A is still in set A.

Since we're told that $(x, y) \in A$ and $(z, w) \in A$ we know $y, w \neq 0$ so $wy \neq 0$. There is no constraint placed on the $xw + zy$ term so the $\diamond$-addition of two elements of A is still in set A, such as $(xw + zy, wy)$. ☒

3. Find an element $(a, b) \in A$ s.t. $(a, b) \diamond (x, y) = (x, y)$ for every $(x, y) \in A$

$y \neq 0$ so we try $(0, 1)$

$(0, 1) \diamond (x, y) = (0(y) + 1(x), y(1)) = (x, y)$ ☒

4. This "new addition" probably looks odd, but have you seen it before. What is it?