The Properties of Squashing Factor Q

Chen, Jun (陈俊)
University of Science and Technology of China
University of Potsdam
el2718@mail.ustc.edu.cn

update time: June 19, 2024

1 Values of Q are same at two foot points of a field line

Here presents a simple proof of the section 3.1-3.3 of [Titov, Hornig, and Démoulin 2002], required by Prof. Hu, Youqiu (胡友秋) and done in 2014.

1.1 Jacobian matrix is inversed at mapping point

[Titov, Hornig, and Démoulin 2002] defines a mapping at photosphere for (x_+, y_+) in the region where $B_z > 0$ to the region where $B_z < 0$ by magnetic field line:

$$\underset{+-}{\Pi} : (x_+, y_+) \rightarrow (x_-, y_-)$$

The Jacobian matrix at (x_+, y_+) (Equation (2) of [Titov, Hornig, and Démoulin 2002]) is

$$\underset{+-}{D} = \begin{pmatrix} \dfrac{\partial x_-}{\partial x_+} & \dfrac{\partial x_-}{\partial y_+} \\ \dfrac{\partial y_-}{\partial x_+} & \dfrac{\partial y_-}{\partial y_+} \end{pmatrix}, \tag{1}$$

and the Jacobian matrix at (x_-, y_-) is

$$\underset{-+}{D} = \begin{pmatrix} \dfrac{\partial x_+}{\partial x_-} & \dfrac{\partial x_+}{\partial y_-} \\ \dfrac{\partial y_+}{\partial x_-} & \dfrac{\partial y_+}{\partial y_-} \end{pmatrix}. \tag{2}$$

Considering the relation between the differential elements $\mathrm{d}x_-, \mathrm{d}y_-, \mathrm{d}x_+, \mathrm{d}y_+$, we have

$$\begin{pmatrix} \mathrm{d}x_- \\ \mathrm{d}y_- \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_-}{\partial x_+} & \dfrac{\partial x_-}{\partial y_+} \\ \dfrac{\partial y_-}{\partial x_+} & \dfrac{\partial y_-}{\partial y_+} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_+ \\ \mathrm{d}y_+ \end{pmatrix}, \text{ and} \tag{3}$$

$$\begin{pmatrix} \mathrm{d}x_+ \\ \mathrm{d}y_+ \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_+}{\partial x_-} & \dfrac{\partial x_+}{\partial y_-} \\ \dfrac{\partial y_+}{\partial x_-} & \dfrac{\partial y_+}{\partial y_-} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_- \\ \mathrm{d}y_- \end{pmatrix}. \tag{4}$$

So $\underset{+-}{D}, \underset{-+}{D}$ are inverse matrixes of each other,

$$\underset{+-}{D} = \left(\underset{-+}{D}\right)^{-1}. \tag{5}$$

Then their determinants are reciprocal of each other,

$$\text{Det}\,\underset{+-}{D} = 1\,/\,\text{Det}\,\underset{-+}{D}\,. \tag{6}$$

Also see this from [inverse of Jacobian matrix].

1.2 A property of a 2×2 matrix

For a 2×2 matrix, if its determinant isn't 0, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\text{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{7}$$

Because

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a\,d-b\,c & 0 \\ 0 & a\,d-b\,c \end{pmatrix}$$

Also see this from [inverse of 2×2 matrix].

1.3 Proof $Q\,(x_+, y_+) = Q\,(x_-, y_-)$

From Equation (24) of [Titov, Hornig, and Démoulin 2002], Q at (x_+, y_+) is

$$Q\,(x_+, y_+) = \frac{\left(\frac{\partial x_-}{\partial x_+}\right)^2 + \left(\frac{\partial x_-}{\partial y_+}\right)^2 + \left(\frac{\partial y_-}{\partial x_+}\right)^2 + \left(\frac{\partial y_-}{\partial y_+}\right)^2}{\left|\text{Det}\,\underset{+-}{D}\right|}. \tag{8}$$

By Equations (5) (7), the numerator of Equation (8) is

$$\left[\left(\frac{\partial x_+}{\partial x_-}\right)^2 + \left(\frac{\partial x_+}{\partial y_-}\right)^2 + \left(\frac{\partial y_+}{\partial x_-}\right)^2 + \left(\frac{\partial y_+}{\partial y_-}\right)^2\right] \bigg/ \left(\text{Det}\,\underset{-+}{D}\right)^2,$$

and we substitute Equations (6) to Equation (8)

$$Q\,(x_+, y_+) = \frac{\left(\frac{\partial x_+}{\partial x_-}\right)^2 + \left(\frac{\partial x_+}{\partial y_-}\right)^2 + \left(\frac{\partial y_+}{\partial x_-}\right)^2 + \left(\frac{\partial y_+}{\partial y_-}\right)^2}{\left|\text{Det}\,\underset{-+}{D}\right|},$$

that is

$$Q\,(x_+, y_+) = Q\,(x_-, y_-)\,. \tag{9}$$

Then we can extend the definition of Q to 3 dimensional space by $\mathbf{B} \cdot \nabla Q = 0$: Q are same along a field line. Therefore [code of Q] can use method 3 of [Pariat and Démoulin 2012] to calculate the Q map at a cross section.

2 The minimum value of Q is 2

Take a, b, c, d of Equation (7) to denote 4 elements of Equation (1),

$$Q = \frac{a^2 + b^2 + c^2 + d^2}{|a\,d - b\,c|}.$$

If $ad - bc > 0$, then Equation (8) can be rewrote as

$$Q = \frac{(a-d)^2 + (b+c)^2 + 2\,a\,d - 2\,b\,c}{a\,d - b\,c} = \frac{(a-d)^2 + (b+c)^2}{a\,d - b\,c} + 2 \geq 2. \tag{10}$$

If $x_- = x_+ + constant1, y_- = y_+ + constant2$, then $a = d = 1$, $b = c = 0$, $Q = 2$, so Q can reach 2 at this case. One can see similar case for $a\,d - b\,c < 0$.

3 Replace $|\mathrm{Det}\,\underset{+-}{D}|$ by $|B_{z+}/B_{z-}|$

Similar to Equation (27) of [Titov, Hornig, and Démoulin 2002], from $\nabla \cdot \mathbf{B} = 0$,

$$|B_{z-}|\,\mathrm{d}x_-\,\mathrm{d}y_- = |B_{z+}|\,\mathrm{d}x_+\,\mathrm{d}y_+,$$

Equation (3) gives

$$\mathrm{d}x_-\,\mathrm{d}y_- = |\mathrm{Det}\,\underset{+-}{D}|\,\mathrm{d}x_+\,\mathrm{d}y_+,$$

therefore

$$|\mathrm{Det}\,\underset{+-}{D}| = |B_{z+}/B_{z-}|. \tag{11}$$

4 Q's value depends on the planes for mapping

Set $\mathbf{B} = (x,\, 2y,\, -3z)$, which satisfies $\nabla \cdot \mathbf{B} = 0$. From equations of field lines

$$\frac{d\,x}{x} = \frac{d\,y}{2\,y} = \frac{d\,z}{-3\,z},$$

therefore

$$x^3\,z = c_1,$$
$$y^{3/2}\,z = c_2,$$

a specific set of c_1, c_2 will result a specific field line. Mapping $(x_1,\, y_1)$ on the plane $z = 1$ to $(x_h,\, y_h)$ on the plane $z = h$:

$$x_h = x_1\,h^{-1/3},$$
$$y_h = y_1\,h^{-2/3}.$$

The Jacobian Matrix is

$$\frac{\partial(x_h, y_h)}{\partial(x_1, y_1)} = \begin{pmatrix} h^{-1/3} & 0 \\ 0 & h^{-2/3} \end{pmatrix}.$$

$$Q = h^{-1/3} + h^{1/3},$$

depends on h. This is the reason we mark the boundary where the foot point of field line locate on in [code of Q].

References

[Titov, Hornig, and Démoulin 2002] Titov, V.S., Hornig, G., and Démoulin, P.: 2002, *Journal of Geophysical Research (Space Physics)* **107**, 1164

[inverse of Jacobian matrix] https://math.stackexchange.com/questions/2956102/show-that-the-product-of-the-jacobian-and-the-inverse-jacobian-is-1

[inverse of 2×2 matrix] https://www.mathcentre.ac.uk/resources/uploaded/sigma-matrices7-2009-1.pdf

[code of Q] https://github.com/el2718/FastQSL; https://github.com/peijin94/FastQSL

[Pariat and Démoulin 2012] Pariat, E. and Démoulin, P.: 2012, *Astronomy and Astrophysics* **541**, A78.