# Numerical solution of tracing magnetic field line

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## Field line equation

A magnetic field line can be described by a vector function $\boldsymbol{r}(s)$, where $s$ is the coordinate on $\boldsymbol{r}$ and has the unit of length : the length of the segment $\boldsymbol{r}(s1) \rightarrow \boldsymbol{r}(s2)$ is $|s2 - s1|$ for any $s1, s2$.

Based on

$$\mathrm{d}\boldsymbol{r} \parallel \boldsymbol{B}, \tag{1}$$

the field line equation is

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\boldsymbol{B}}{B}. \tag{2}$$

In Cartesian coordinates, $\boldsymbol{r}(s) = (x(s),\, y(s),\, z(s))$, then equation (2) becomes:

$$\frac{\mathrm{d}\,(x,\, y,\, z)}{\mathrm{d}s} = \frac{(B_x,\, B_y,\, B_z)}{B}. \tag{3}$$

If $\mathrm{d}x \neq 0$, the independent variable $s$ can be changed to $x$: $\boldsymbol{r}(x) = (x,\, y(x),\, z(x))$, then equation (3) is equivalent to

$$\frac{\mathrm{d}\,(y,\, z)}{\mathrm{d}x} = \frac{(B_y,\, B_z)}{B_x} \tag{4}$$

## Numerical solution

In [code of Q], equation (3) is integrated by [4th order Runge-Kutta method] :

$$\begin{aligned}
\boldsymbol{k}_1 &= \boldsymbol{B}(\boldsymbol{r}_n) \,/\, B(\boldsymbol{r}_n) \\
\boldsymbol{k}_2 &= \boldsymbol{B}(\boldsymbol{r}_n + \mathrm{d}t\, \boldsymbol{k}_1/2) \,/\, B(\boldsymbol{r}_n + \mathrm{d}t\, \boldsymbol{k}_1/2) \\
\boldsymbol{k}_3 &= \boldsymbol{B}(\boldsymbol{r}_n + \mathrm{d}t\, \boldsymbol{k}_2/2) \,/\, B(\boldsymbol{r}_n + \mathrm{d}t\, \boldsymbol{k}_2/2) \\
\boldsymbol{k}_4 &= \boldsymbol{B}(\boldsymbol{r}_n + \mathrm{d}t\, \boldsymbol{k}_3) \,/\, B(\boldsymbol{r}_n + \mathrm{d}t\, \boldsymbol{k}_3) \\
\boldsymbol{r}_{n+1} &= \boldsymbol{r}_n + \frac{\mathrm{d}t}{6}\,(\boldsymbol{k}_1 + 2\,\boldsymbol{k}_2 + 2\,\boldsymbol{k}_3 + \boldsymbol{k}_4)
\end{aligned}$$

And $\boldsymbol{B}(\boldsymbol{r})$ is achieved by [trilinear interpolation] in the data cube of $\boldsymbol{B}$.

At the boundary of $x = 0$, if $\boldsymbol{r}_{n+1}$ is outside of the box and $\boldsymbol{r}_n$ is inside, we take $\mathrm{d}x = 0 - x_n$, and integrate equation (4) by [4th order Runge-Kutta method], then $x_{n+1}$ is identical to 0. [code of Q] does similar treatment to other boundaries.

## References

[code of Q] http://staff.ustc.edu.cn/~rliu/qfactor.html

[4th order Runge-Kutta method] https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods

[trilinear interpolation] https://en.wikipedia.org/wiki/Trilinear_interpolation