无量纲化守恒形式的 MHD 方程组

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前言

本文源自于 Bernhard Kliem 要求的一个他所有学生都要完成的作业，完成于 2018 年春。其中用到的张量知识见 [基矢量与张量]。

1 磁流体动力学方程组

定义随体导数 $\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$, MHD 方程组的随体形式为

$$\frac{\mathrm{D}\,\rho}{\mathrm{D}t} = -\rho\,\nabla\cdot\boldsymbol{u} \tag{1}$$

$$\rho\frac{\mathrm{D}\,\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \boldsymbol{J}\times\boldsymbol{B} + \nabla\cdot(\rho\,\nu\,(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \frac{2}{3}\boldsymbol{I}\,\nabla\cdot\boldsymbol{u})) \tag{2}$$

$$\frac{\partial\,\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) - \nabla\times(\eta\,\boldsymbol{J}) \tag{3}$$

$$\rho\frac{\mathrm{D}\,e}{\mathrm{D}t} = -p\,\nabla\cdot\boldsymbol{u} + \eta\,J^2 + 2\,\rho\,\nu\,(\frac{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T}{2}) : (\frac{\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T}{2}) - \frac{2}{3}\rho\,\nu\,(\nabla\cdot\boldsymbol{u})^2 \tag{4}$$

并注意到

$$\nabla\cdot\boldsymbol{B} = 0, \tag{5}$$

$$\boldsymbol{J} \equiv \frac{1}{\mu_0}\nabla\times\boldsymbol{B}, \tag{6}$$

$$p = (\gamma - 1)\,\rho\,e. \tag{7}$$

2 守恒形式

对于一个守恒量 ξ, 在某一确定空间内总量的变化为边界的通量，即

$$\frac{\partial}{\partial t}\int_V \xi dV = -\int_{\partial V} \boldsymbol{f}(\xi)\cdot \boldsymbol{dS}, \tag{8}$$

对应的微分形式就是

$$\frac{\partial\,\xi}{\partial t} + \nabla\cdot\boldsymbol{f}(\xi) = 0\,. \tag{9}$$

所以称式(9)为守恒形式。

2.1 连续性方程

式(1) 可以写成

$$\frac{\partial\,\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho + \rho\nabla\cdot\boldsymbol{u} = 0,$$

即

$$\frac{\partial\,\rho}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{u}) = 0. \tag{10}$$

2.2 动量方程

结合式(1)，式(2)的等号左边项

$$\begin{aligned}\rho\frac{\mathrm{D}\,\boldsymbol{u}}{\mathrm{D}t} &= \frac{\mathrm{D}\,\rho\,\boldsymbol{u}}{\mathrm{D}t} - \boldsymbol{u}\,\frac{\mathrm{D}\,\rho}{\mathrm{D}t}\\ &= \frac{\partial\,\rho\,\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla(\rho\,\boldsymbol{u}) + \rho\,\boldsymbol{u}\,\nabla\cdot\boldsymbol{u}\\ &= \frac{\partial\,\rho\,\boldsymbol{u}}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{u}\,\boldsymbol{u}).\end{aligned} \tag{11}$$

式(2)的等号右边第一项

$$\nabla p = \nabla\cdot(p\,\boldsymbol{I}). \tag{12}$$

结合式(5)，式(2)的等号右边第二项

$$\begin{aligned}\boldsymbol{J}\times\boldsymbol{B} &= \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}\\ &= \frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla\boldsymbol{B} - (\nabla\boldsymbol{B})\cdot\boldsymbol{B})\\ &= \frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla\boldsymbol{B} + (\nabla\cdot\boldsymbol{B})\boldsymbol{B} - \nabla\frac{B^2}{2})\\ &= \frac{1}{\mu_0}(\nabla\cdot(\boldsymbol{B}\boldsymbol{B} - \frac{B^2}{2}\boldsymbol{I})).\end{aligned} \tag{13}$$

结合式(2)(11)(12)(13)，且记粘性项

$$\boldsymbol{T} := \rho\,\nu\,(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \frac{2}{3}\boldsymbol{I}\ \nabla\cdot\boldsymbol{u}), \tag{14}$$

得到

$$\frac{\partial\,\rho\,\boldsymbol{u}}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{u}\,\boldsymbol{u} + p\,\boldsymbol{I} - \frac{1}{\mu_0}(\boldsymbol{B}\boldsymbol{B} - \frac{B^2}{2}\boldsymbol{I}) - \boldsymbol{T}) = 0 \tag{15}$$

2.3 磁感应方程

式(3)的等号右边第一项

$$\begin{aligned}\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) &= \boldsymbol{B}\cdot\nabla\boldsymbol{u} + (\nabla\cdot\boldsymbol{B})\,\boldsymbol{u} - (\nabla\cdot\boldsymbol{u})\,\boldsymbol{B} - \boldsymbol{u}\cdot\nabla\boldsymbol{B}\\ &= \nabla\cdot(\boldsymbol{B}\,\boldsymbol{u} - \boldsymbol{u}\,\boldsymbol{B}).\end{aligned} \tag{16}$$

式(3)的等号右边第二项

$$\nabla\times(\eta\,\boldsymbol{J}) = \nabla\cdot(-\eta\,\boldsymbol{\varepsilon}\cdot\boldsymbol{J}). \tag{17}$$

结合式(3)(16)(17)就有

$$\frac{\partial\,\boldsymbol{B}}{\partial t} + \nabla\cdot(\boldsymbol{u}\,\boldsymbol{B} - \boldsymbol{B}\,\boldsymbol{u} - \eta\ \boldsymbol{\varepsilon}\cdot\boldsymbol{J}) = 0. \tag{18}$$

2.4 能量方程

结合式(1)(7), 式(4)的等号左边项

$$\begin{aligned}\rho\,\frac{\mathrm{D}\,e}{\mathrm{D}t} &= \frac{\mathrm{D}\,(\rho\,e)}{\mathrm{D}t} - e\,\frac{\mathrm{D}\,\rho}{\mathrm{D}t}\\ &= \frac{1}{\gamma-1}\,(\frac{\partial\,p}{\partial t} + \boldsymbol{u}\cdot\nabla p + p\,\nabla\cdot\boldsymbol{u})\\ &= \frac{1}{\gamma-1}\,(\frac{\partial\,p}{\partial t} + \nabla\cdot(\boldsymbol{u}\,p)).\end{aligned} \tag{19}$$

式(4)的等号右边第一项

$$p\,\nabla\cdot\boldsymbol{u} = \nabla\cdot(\boldsymbol{u}\,p) - \boldsymbol{u}\cdot\nabla p. \tag{20}$$

式(4)的等号右边第二项由式(15)可得

$$\begin{aligned}-\boldsymbol{u}\cdot\nabla p = \boldsymbol{u}\cdot[\frac{\partial\,\rho\,\boldsymbol{u}}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{u}\,\boldsymbol{u})] - \boldsymbol{u}\cdot[\frac{1}{\mu_0}(\boldsymbol{B}\boldsymbol{B} - \frac{B^2}{2}\boldsymbol{I})] - \\ \boldsymbol{u}\cdot\{\nabla\cdot[\rho\,\nu\,(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \frac{2}{3}\boldsymbol{I}\ \nabla\cdot\boldsymbol{u})]\}.\end{aligned} \tag{21}$$

结合式(10), 式(21)的等号右边第一项的第一部分

$$\begin{aligned}\boldsymbol{u}\cdot\frac{\partial\,\rho\,\boldsymbol{u}}{\partial t}&=\boldsymbol{u}\cdot(\rho\frac{\partial\,\boldsymbol{u}}{\partial t}+\boldsymbol{u}\frac{\partial\,\rho}{\partial t})\\&=\frac{\rho}{2}\frac{\partial\,u^2}{\partial t}+\frac{u^2}{2}\frac{\partial\,\rho}{\partial t}+\frac{u^2}{2}\frac{\partial\,\rho}{\partial t}\\&=\frac{1}{2}[\frac{\partial\,\rho\,u^2}{\partial t}-u^2\nabla\cdot(\rho\,\boldsymbol{u})]\end{aligned}\tag{22}$$

式(21)的等号右边第一项的第二部分

$$\begin{aligned}\boldsymbol{u}\cdot[\nabla\cdot(\rho\,\boldsymbol{u}\,\boldsymbol{u})]&=\boldsymbol{u}\cdot[\nabla\cdot(\rho\,\boldsymbol{u})\,\boldsymbol{u}+\rho\,\boldsymbol{u}\cdot\nabla\boldsymbol{u}]\\&=\nabla\cdot(\rho\,\boldsymbol{u})\,u^2+\rho\,\boldsymbol{u}\cdot[(\nabla\boldsymbol{u})\cdot\boldsymbol{u}]\\&=\nabla\cdot(\rho\,\boldsymbol{u})\,u^2+\rho\,\boldsymbol{u}\cdot\nabla\frac{u^2}{2}\end{aligned}\tag{23}$$

那么式(21)的等号右边第一项就是

$$\begin{aligned}\boldsymbol{u}\cdot[\frac{\partial\,\rho\,\boldsymbol{u}}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{u}\,\boldsymbol{u})]&=\frac{1}{2}(\frac{\partial\,\rho\,u^2}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{u})\,u^2+\rho\,\boldsymbol{u}\cdot\nabla u^2)\\&=\frac{1}{2}[\frac{\partial\,\rho\,u^2}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{u}\,u^2)]\end{aligned}\tag{24}$$

结合式(17)(18)

$$\begin{aligned}\nabla\cdot[(\boldsymbol{u}\,\boldsymbol{B}-\boldsymbol{B}\,\boldsymbol{u})\cdot\boldsymbol{B}]&=\nabla\cdot(\boldsymbol{u}\,\boldsymbol{B}-\boldsymbol{B}\,\boldsymbol{u})\cdot\boldsymbol{B}+\boldsymbol{u}\cdot(\nabla\boldsymbol{B})\cdot\boldsymbol{B}-\boldsymbol{B}\cdot(\nabla\boldsymbol{B})\cdot\boldsymbol{u}\\&=-[\frac{\partial\,\boldsymbol{B}}{\partial t}+\nabla\times(\eta\,\boldsymbol{J})]\cdot\boldsymbol{B}+\boldsymbol{u}\cdot\nabla\frac{B^2}{2}-\nabla\cdot(\boldsymbol{B}\boldsymbol{B})\cdot\boldsymbol{u}\\&=-[\frac{\partial\,(B^2/2)}{\partial t}+\nabla\cdot(\eta\,\boldsymbol{J}\times\boldsymbol{B})+\eta\,(\nabla\times\boldsymbol{B})\cdot\boldsymbol{J}]-\boldsymbol{u}\cdot[\nabla\cdot(\boldsymbol{B}\boldsymbol{B}-\frac{B^2}{2}\boldsymbol{I})]\\&=-[\frac{\partial\,(B^2/2)}{\partial t}+\nabla\cdot(\eta\,\boldsymbol{J}\times\boldsymbol{B})+\mu_0\,\eta\,J^2]+\boldsymbol{u}\cdot[\nabla\cdot(\boldsymbol{B}\boldsymbol{B}-\frac{B^2}{2}\boldsymbol{I})]\end{aligned}$$

可得式(21)的等号右边第二项

$$-\frac{1}{\mu_0}\boldsymbol{u}\cdot[\nabla\cdot(\boldsymbol{B}\boldsymbol{B}-\frac{B^2}{2}\boldsymbol{I})]=\frac{1}{\mu_0}\nabla\cdot[(\boldsymbol{u}\,\boldsymbol{B}-\boldsymbol{B}\,\boldsymbol{u})\cdot\boldsymbol{B}]+\frac{\partial\,(B^2/(2\,\mu_0))}{\partial t}+\nabla\cdot(\frac{\eta}{\mu_0}\boldsymbol{J}\times\boldsymbol{B})+\eta\,J^2\tag{25}$$

式(21)的等号右边第三项

$$\begin{aligned}&-\boldsymbol{u}\cdot\{\nabla\cdot[\rho\,\nu\,(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T-\frac{2}{3}\boldsymbol{I}\ \nabla\cdot\boldsymbol{u})]\}\\=&-\nabla\cdot[\boldsymbol{u}\cdot\rho\,\nu\,(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T-\frac{2}{3}\boldsymbol{I}\ \nabla\cdot\boldsymbol{u})]+\nabla\boldsymbol{u}\cdot\cdot[\rho\,\nu\,(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T-\frac{2}{3}\boldsymbol{I}\ \nabla\cdot\boldsymbol{u})]\\=&-\nabla\cdot[\rho\,\nu\,(\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla\frac{u^2}{2}-\frac{2}{3}\boldsymbol{u}\ \nabla\cdot\boldsymbol{u})]+\rho\,\nu[\nabla\boldsymbol{u}\cdot\cdot\nabla\boldsymbol{u}+\nabla\boldsymbol{u}:\nabla\boldsymbol{u}-\frac{2}{3}(\nabla\cdot\boldsymbol{u})^2]\end{aligned}\tag{26}$$

式(4)的等号右边第三项

$$
\begin{aligned}
&2\,\rho\,\nu\,(\frac{\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^T}{2}):(\frac{\nabla \boldsymbol{u}+(\nabla \boldsymbol{u})^T}{2})\\
=&\frac{1}{2}\,\rho\,\nu\,(\nabla \boldsymbol{u}:\nabla \boldsymbol{u}+2\nabla \boldsymbol{u}:(\nabla \boldsymbol{u})^T+(\nabla \boldsymbol{u})^T:(\nabla \boldsymbol{u})^T)\\
=&\frac{1}{2}\,\rho\,\nu\,(\nabla \boldsymbol{u}:\nabla \boldsymbol{u}+2\nabla \boldsymbol{u}\cdot\cdot\nabla \boldsymbol{u}+\nabla \boldsymbol{u}:\nabla \boldsymbol{u})\\
=&\rho\,\nu\,(\nabla \boldsymbol{u}:\nabla \boldsymbol{u}+\nabla \boldsymbol{u}\cdot\cdot\nabla \boldsymbol{u})
\end{aligned}
\tag{27}
$$

结合式 (4)(19)(20)(21)(24)(25)(26)(27), 且记总能量密度

$$E:=\frac{p}{\gamma-1}+\frac{\rho\,u^2}{2}+\frac{B^2}{2\,\mu_0}, \tag{28}$$

最终得到

$$\frac{\partial E}{\partial t}+\nabla\cdot[\boldsymbol{u}\,(E+p-\frac{B^2}{2\,\mu_0})-\boldsymbol{u}\cdot\boldsymbol{T}+\frac{1}{\mu_0}(\boldsymbol{u}\,\boldsymbol{B}-\boldsymbol{B}\,\boldsymbol{u}-\eta\,\boldsymbol{\varepsilon}\cdot\boldsymbol{J})\cdot\boldsymbol{B}]=0 \tag{29}$$

3 无量纲化

对于基本量，取特征长度为 L, 特征磁场强度为 B_0, 特征电阻率为 η_0, 就有无量纲的

$$\nabla'=L\,\nabla, \tag{30}$$

$$\rho'=\rho/\rho_0, \tag{31}$$

$$\boldsymbol{B}'=\boldsymbol{B}/B_0, \tag{32}$$

$$\eta'=\eta/\eta_0, \tag{33}$$

特征压强就是 $\frac{B_0^2}{2\,\mu_0}$, 就有无量纲的

$$p'=p/(\frac{B_0^2}{2\,\mu_0}), \tag{34}$$

并定义特征 Alfvén 速度, Reynolds 数, 磁 Reynolds 数

$$V_A:=\frac{B_0}{\sqrt{\mu_0\,\rho_0}}, \tag{35}$$

$$R_e:=\frac{\rho_0\,V_A\,L}{\mu}=\frac{V_A\,L}{\nu}, \tag{36}$$

$$R_m:=\mu_0\,V_A\,L/\eta_0, \tag{37}$$

特征时间就是 $\frac{L}{V_A}$，可以导出无量纲的

$$\boldsymbol{u}'=\boldsymbol{u}/V_A, \tag{38}$$

$$t' = t/(\frac{L}{V_A}). \tag{39}$$

还可以定义无量纲的

$$\boldsymbol{J}' := \nabla' \times \boldsymbol{B}', \tag{40}$$

$$E' := \frac{p'}{\gamma - 1} + \rho' u'^2 + B'^2, \tag{41}$$

$$\boldsymbol{T}' := \rho'(\nabla' \boldsymbol{u}' + (\nabla' \boldsymbol{u}')^T - \frac{2}{3} \boldsymbol{I}\ \nabla' \cdot \boldsymbol{u}'). \tag{42}$$

那么，式(10) 变成

$$\frac{\partial \rho'}{\partial t'} + \nabla' \cdot (\rho' \boldsymbol{u}') = 0. \tag{43}$$

式(15) 变成

$$\frac{\partial \rho' \boldsymbol{u}'}{\partial t'} + \nabla' \cdot [\rho' \boldsymbol{u}' \boldsymbol{u}' + \frac{1}{2} p' \boldsymbol{I} - (\boldsymbol{B}' \boldsymbol{B}' - \frac{B'^2}{2} \boldsymbol{I}) - R_e^{-1} \boldsymbol{T}'] = 0. \tag{44}$$

式 (18) 变成

$$\frac{\partial \boldsymbol{B}'}{\partial t'} + \nabla' \cdot (\boldsymbol{u}' \boldsymbol{B}' - \boldsymbol{B}' \boldsymbol{u}' - R_m^{-1} \eta' \ \boldsymbol{\varepsilon} \cdot \boldsymbol{J}') = 0. \tag{45}$$

式(29) 变成

$$\frac{\partial E'}{\partial t'} + \nabla' \cdot [\boldsymbol{u}' (E' + p' - B'^2) + 2 (\boldsymbol{u}' \boldsymbol{B}' - \boldsymbol{B}' \boldsymbol{u}' - R_m^{-1} \eta' \boldsymbol{\varepsilon} \cdot \boldsymbol{J}') \cdot \boldsymbol{B}' - 2 R_e^{-1} \boldsymbol{u}' \cdot \boldsymbol{T}'] = 0. \tag{46}$$

参考材料

[基矢量与张量] https://github.com/el2718/thoughts/releases/tag/thoughts